

ON SPECIAL IDENTITIES FOR DIALGEBRAS

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ABSTRACT. For every variety of algebras over a field, there is a natural definition of a corresponding variety of dialgebras (Loday-type algebras). In particular, Lie dialgebras are equivalent to Leibniz algebras. We use an approach based on the notion of an operad to study the problem of finding special identities for dialgebras. It is proved that all polylinear special identities for dialgebras can be obtained from special identities for corresponding algebras by means of a simple procedure. A particular case of this result confirms the conjecture by M. Bremner, R. Felipe, and J. Sanchez-Ortega, arXiv:1108.0586.

1. INTRODUCTION

The notion of a Leibniz algebra appeared first in [2] and later independently in [14] gave rise to a series of research devoted to the theory of dialgebras. By definition, a (left) Leibniz algebra is a linear space with a bilinear operation $[\cdot, \cdot]$ which satisfies the Jacobi identity in the form $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$, i.e., the operator of left multiplication $[x, \cdot]$ is a derivation. This is one of the most studied noncommutative analogues of Lie algebras.

Various classes of dialgebras appeared in the literature since they are related to Leibniz algebras in the same way as the corresponding classes of ordinary algebras are related to Lie algebras. Associative dialgebras were introduced in [16] as analogues of associative enveloping algebras for Leibniz algebras, alternative dialgebras appeared in [13] in the study of universal central extensions for Leibniz algebras, Jordan dialgebras (first under the name of quasi-Jordan algebras) were proposed in [19], see also [3] and [11]. All dialgebras of these classes are linear spaces equipped by two bilinear operations \vdash and \dashv such that

$$(1) \quad (x \dashv y) \vdash z = (x \vdash y) \vdash z, \quad x \dashv (y \vdash z) = x \dashv (y \dashv z).$$

These identities are common for associative, alternative, Jordan dialgebras mentioned above, and they also hold for Leibniz algebras provided that $a \vdash b = [a, b]$, $a \dashv b = -[b, a]$. Other defining identities of these varieties initially appeared from a posteriori considerations motivated by relations with Leibniz algebras. For example, a dialgebra is associative if in addition to (1) the following identities hold

$$\begin{aligned} x \vdash (y \vdash z) &= (x \vdash y) \vdash z, & x \vdash (y \dashv z) &= (x \vdash y) \dashv z, \\ x \dashv (y \vdash z) &= (x \dashv y) \vdash z. \end{aligned}$$

Then the same space with respect to the new operation $[a, b] = a \vdash b - b \dashv a$ is a Leibniz algebra. A systematical study of this relation between Leibniz algebras and associative dialgebras may be found in [15].

An idea of a more conceptual approach to the definition of what should be called a dialgebra associated with a given variety \mathfrak{P} of ordinary algebras was proposed in [7] in the case of associative algebras: It was shown that the operad governing the

variety of associative dialgebras in the sense of [16] coincides with the Hadamard product $\text{As} \otimes \text{Perm}$, where As is the operad governing the variety of associative algebras and Perm is the operad governing associative algebras satisfying the left commutativity relation $(xy)z - (yx)z = 0$.

For an arbitrary variety \mathfrak{P} of ordinary algebras with one binary operation (governed by an operad \mathcal{P}), the algorithm proposed in [11] and [17] allows to deduce the defining identities for the class of (di-)algebras governed by the operad $\mathcal{P} \otimes \text{Perm}$ starting with the defining identities of \mathfrak{P} . In [4], this algorithm was generalized to the case of arbitrary varieties of algebras of any type (i.e., linear spaces with a family of polylinear operations of arbitrary arity). In this note, we will show that this generalized algorithm also leads to the class of $\mathcal{P} \otimes \text{Perm}$ -algebras. This is why we denote by $\mathcal{P} \otimes \text{Perm}$ by $\text{di}\mathcal{P}$.

This fact allows to consider a series of questions devoted to elementary properties and relations between various classes of dialgebras from a unified point of view. In particular, a morphism of operads $\omega : \mathcal{P} \rightarrow \mathcal{R}$ always gives rise to a functor from the category of \mathcal{R} -algebras to the category of \mathcal{P} -algebras. So are the well-known functors:

\mathcal{R}	\mathcal{P}	ω
Associative	Lie	$x_1x_2 \mapsto x_1x_2 - x_2x_1$
Associative	Jordan	$x_1x_2 \mapsto x_1x_2 + x_2x_1$
Alternative	Jordan	$x_1x_2 \mapsto x_1x_2 + x_2x_1$
Alternative	Mal'cev	$x_1x_2 \mapsto x_1x_2 - x_2x_1$
Associative	Jordan triple system	$\langle x_1, x_2, x_3 \rangle \mapsto x_1x_2x_3 + x_3x_2x_1$
Jordan	Jordan triple system	$\langle x_1, x_2, x_3 \rangle \mapsto (x_1x_2)x_3 - (x_1x_3)x_2 + x_1(x_2x_3)$

For each triple $(\mathcal{P}, \mathcal{R}, \omega)$ as above the following *speciality problem* makes sense: Whether the variety generated by all those \mathcal{P} -algebras obtained from \mathcal{R} -algebras coincides with the class of all \mathcal{P} -algebras? If no, what are the identities separating these classes (special identities)? The same question is actual for dialgebras: The corresponding varieties of $\text{di}\mathcal{R}$ - and $\text{di}\mathcal{P}$ -algebras are related by a functor raising from the morphism $\omega \otimes \text{id}$ of operads.

The purpose of this note is to show that the speciality problem for dialgebras raising from the triple $(\text{di}\mathcal{P}, \text{di}\mathcal{R}, \omega \otimes \text{id})$ can always be solved modulo the same problem for ordinary algebras.

2. THE BSO ALGORITHM

Let us start with the construction from [4], assuming the base field \mathbb{k} is of zero characteristic.

Let A be an associative algebra over \mathbb{k} equipped by new n -ary operation

$$(2) \quad \omega(x_1, \dots, x_n) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \dots x_{\sigma(n)},$$

where $\alpha_\sigma \in \mathbb{k}$.

Choose an index $i \in \{1, \dots, n\}$. One may rewrite (2) as

$$\omega(x_1, \dots, x_n) = \sum_{j=1}^n \sum_{S_n^{ji}} \alpha_\sigma x_{\sigma(1)} \dots x_{\sigma(j-1)} x_i x_{\sigma(j+1)} \dots x_{\sigma(n)},$$

where S_n^{ji} — is a set of permutations such that $\sigma(j) = i$.

Denote by $\text{Id}(\omega)$ the set of all polylinear identities satisfied for all n -ary algebras obtained in this way from associative ones.

Starting from the identities $\text{Id}(\omega)$, one may canonically construct a set of identities $\text{Id}(\omega)^{(2)}$ of type $\{\omega_1, \dots, \omega_n\}$, where each ω_i is an n -ary operation. The algorithm of such a construction was described in [4] (as a KP algorithm), and also in Section 6.

On the other hand, consider the following operations on an associative dialgebra D :

$$\omega_i(x_1, \dots, x_n) = \sum_{j=1}^n \sum_{S_n^{ji}} \alpha_\sigma x_{\sigma(1)} \vdash \dots \vdash x_{\sigma(j-1)} \vdash x_i \dashv x_{\sigma(j+1)} \dashv \dots \dashv x_{\sigma(n)},$$

$i = 1, \dots, n$ (the bracketing is not essential here). The family of n -ary operations $\omega_1, \dots, \omega_n$ obtained are denoted by $\text{BSO}(\omega)$. Let $\text{Id}(\text{BSO}(\omega))$ stand for the set of all polylinear identities satisfied for all algebras with n -ary operations $\text{BSO}(\omega)$ obtained in this way from associative dialgebras.

Problem 1 ([4]). Let $\text{char } \mathbb{k} = 0$. Prove that for every choice of ω we have $\text{Id}(\text{BSO}(\omega)) = \text{Id}(\omega)^{(2)}$.

Next, suppose $\text{char } \mathbb{k} = p > 0$. Then the relation in Problem 1 is not valid in general, but it is reasonable to state

Problem 2 ([4]). For $\text{char } \mathbb{k} = p > 0$ and $d < p$, prove that for every choice of ω we have $\text{Id}_d(\text{BSO}(\omega)) = \text{Id}_d(\omega)^{(2)}$, where $\text{Id}_d(\cdot)$ stands for the subset of identities of degree d in $\text{Id}(\cdot)$.

In this paper, we solve these problems.

3. PRELIMINARIES IN OPERADS

In this section, we state the necessary notions of the operad theory following mainly [8], with a particular accent on the operads governing varieties of algebras.

A *language* Ω is a set of functional symbols $\{f_i \mid i \in I\}$ equipped by an *arity function* $\nu : f_i \mapsto n_i \equiv \nu(f_i) \in \mathbb{N}$. An Ω -*algebra* is a linear space A over a base field \mathbb{k} endowed with linear maps $f_i^A : A^{\otimes n_i} \rightarrow A$, $i \in I$ [12]. Below, we will use the term *algebra of type* Ω for an Ω -algebra to avoid confusion. In this paper, we assume $n_i \geq 2$.

Denote by $\mathcal{F}_\Omega\langle X \rangle$ the free algebra of type Ω generated by the countable set $X = \{x_1, x_2, \dots\}$. The linear basis of this algebra consists of all terms of type Ω in variables from X . Let us call such terms *monomials*, their linear combinations (elements of the free algebra) are called *polynomials*.

For every $n \in \mathbb{N}$ consider the space $\mathcal{F}_\Omega(n)$ of all polylinear polynomials of degree n in x_1, \dots, x_n . The composition

$$\gamma_{m_1, \dots, m_n} : \mathcal{F}_\Omega(n) \otimes \mathcal{F}_\Omega(m_1) \otimes \dots \otimes \mathcal{F}_\Omega(m_n) \rightarrow \mathcal{F}_\Omega(m_1 + \dots + m_n)$$

of such maps is naturally defined by the rule

$$\gamma_{m_1, \dots, m_n}(f; g_1, \dots, g_n) = f(g_1(x_1, \dots, x_{m_1}), g_2(x_{m_1+1}, \dots, x_{m_1+m_2}), \dots),$$

where $f(x_1, \dots, x_n) \in \mathcal{F}_\Omega(n)$, $g_i(x_1, \dots, x_{m_i}) \in \mathcal{F}_\Omega(m_i)$, $i = 1, \dots, n$; the result belongs to $\mathcal{F}_\Omega(m_1 + \dots + m_n)$. The simplest term $x_1 \in \mathcal{F}_\Omega(1)$ behaves as an

identity with respect to this composition. Symmetric groups S_n act on $\mathcal{F}_\Omega(n)$ by permutations of variables.

The collection of spaces $\{\mathcal{F}_\Omega(n)\}_{n \in \mathbb{N}}$ together with above-mentioned composition rule, identity element, and S_n -action is a particular case of an operad which is natural to call the *free operad* \mathcal{F}_Ω generated by Ω .

Given two operads \mathcal{P} and \mathcal{R} , a morphism $\alpha : \mathcal{P} \rightarrow \mathcal{R}$ is just a family of S_n -linear maps $\{\alpha(n)\}_{n \in \mathbb{N}}$,

$$\alpha(n) : \mathcal{P}(n) \rightarrow \mathcal{R}(n)$$

preserving the composition and the identity element. The kernel of α is the collection of subspaces (even S_n -submodules) $\text{Ker } \alpha(n) \subseteq \mathcal{P}(n)$, $n \in \mathbb{N}$, which is closed with respect to compositions in the obvious sense. Such a family of subspaces is called an *operad ideal* in \mathcal{P} .

To define a morphism π from \mathcal{F}_Ω to an operad \mathcal{P} it is enough to determine $\pi(n_i)(f_i)$, $f_i = f_i(x_1, \dots, x_{n_i}) \in \mathcal{F}_\Omega(n_i)$, where f_i range through the language Ω , $n_i = \nu(f_i)$. Moreover, every family $g_i \in \mathcal{P}(n_i)$, $i \in I$, defines a unique morphism of operads $\pi : \mathcal{F}_\Omega \rightarrow \mathcal{P}$ such that $\pi(n_i)(f_i) = g_i$.

Every linear space A gives rise to an operad $\mathcal{E}(A)$, the operad of endomorphisms of A . Namely, $\mathcal{E}(A)(n) = \text{Hom}(A^{\otimes n}, A)$, $n \in \mathbb{N}$, compositions and S_n -actions are defined in the ordinary way.

A structure of an algebra of type Ω on a linear space A may be identified with a morphism of operads $\alpha : \mathcal{F}_\Omega \rightarrow \mathcal{E}(A)$ such that $\alpha(n_i)(f_i) = f_i^A$, $i \in I$. Conversely, every morphism $\alpha : \mathcal{F}_\Omega \rightarrow \mathcal{E}(A)$ defines a structure of an algebra on A .

Suppose \mathfrak{P} is a variety of algebras of type Ω defined by polylinear identities (this is a generic case if $\text{char } \mathbb{k} = 0$). Then the following consideration makes sense.

Let $T(\mathfrak{P})$ be the ideal of identities (T-ideal) in $\mathcal{F}_\Omega\langle X \rangle$ corresponding to the variety \mathfrak{P} . Denote $\mathcal{P}(n) = \mathcal{F}_\Omega(n)/(T(\mathfrak{P}) \cap \mathcal{F}_\Omega(n))$, $n \in \mathbb{N}$. The composition rule and S_n -actions are well-defined on the family $\{\mathcal{P}(n)\}_{n \in \mathbb{N}}$, so this collection is also an operad. Such an operad is said to be the *governing operad* for the variety \mathfrak{P} . There exists a natural quotient morphism $\pi : \mathcal{F}_\Omega \rightarrow \mathcal{P}$. If S is a defining family of polylinear identities of the variety \mathfrak{P} then the kernel of π is exactly the operad ideal generated by S in \mathcal{F}_Ω .

Every algebra A from the variety \mathfrak{P} is determined by a composition $\pi \circ \bar{\alpha}$, where $\bar{\alpha}$ is a morphism from \mathcal{P} to $\mathcal{E}(A)$. Thus, A is defined by a morphism of operads $\mathcal{P} \rightarrow \mathcal{E}(A)$. Conversely, every morphism of this kind defines an algebra structure on A , and the algebra obtained belongs to \mathfrak{P} .

In general, given an operad \mathcal{P} , a \mathcal{P} -*algebra* is a pair (A, α) of a linear space A and a morphism of operads $\alpha : \mathcal{P} \rightarrow \mathcal{E}(A)$.

4. CONFORMAL ALGEBRAS

The notion of a conformal algebra was introduced in [10] as tool of vertex operator algebra study. In a more general context, a conformal algebra is a pseudo-algebra over the polynomial algebra $\mathbb{k}[T]$ in one variable [1]. Here we consider the last approach for arbitrary set of operations Ω .

As we have already mentioned, every linear space A gives rise to the operad $\mathcal{E}(A)$. A similar construction exists for left unital modules over a cocommutative bialgebra H . Suppose M is such a module, then denote

$$\mathcal{E}^*(M)(n) = \text{Hom}_{H^{\otimes n}}(M^{\otimes n}, H^{\otimes n} \otimes_H M).$$

Hereinafter, the symbol \otimes without a subscript stands for the tensor product of spaces over the base field. The space $H^{\otimes n}$ is considered as the outer product of regular right H -modules, i.e.,

$$(h_1 \otimes \cdots \otimes h_n) \cdot h = \sum_{(h)} h_1 h_{(1)} \otimes \cdots \otimes h_n h_{(n)},$$

where $\sum_{(h)} h_{(1)} \otimes \cdots \otimes h_{(n)}$ is the value of n -iterated coproduct on h . Compositions

γ_{m_1, \dots, m_n} of such maps and the action of S_n on $\mathcal{E}^*(M)(n)$ were defined in [1], see also [11] (one needs cocommutativity of H to ensure the action of S_n is well-defined).

A *conformal algebra* over H is a pair (C, α) , where C is an H -module as above and $\alpha : \mathcal{F}_\Omega \rightarrow \mathcal{E}^*(C)$ is a morphism of operads. If α splits into $\mathcal{F}_\Omega \xrightarrow{\pi} \mathcal{P} \xrightarrow{\bar{\alpha}} \mathcal{E}^*(C)$ then C is said to be a \mathcal{P} -conformal algebra.

A simple but important example of a conformal algebra may be constructed as follows. Let (A, α) be a \mathcal{P} -algebra. Consider the free H -module $C = H \otimes A$ and define $\beta = \text{Cur } \alpha : \mathcal{P} \rightarrow \mathcal{E}^*(C)$ by the rule

$$\beta(n)(f) : (h_1 \otimes a_1) \otimes \cdots \otimes (h_n \otimes a_n) \mapsto (h_1 \otimes \cdots \otimes h_n) \otimes_H \alpha(f)(a_1 \otimes \cdots \otimes a_n),$$

$f \in \mathcal{P}(n)$, $h_k \in H$, $a_k \in A$. This is a morphism of operads, and the \mathcal{P} -conformal algebra (C, β) obtained is denoted $(\text{Cur } A, \text{Cur } \alpha)$, the *current* conformal algebra over A .

The correspondence $A \mapsto \text{Cur } A$ is a functor from the category of \mathcal{P} -algebras to the category of \mathcal{P} -conformal algebras: Every morphism φ between \mathcal{P} -algebras can be continued by H -linearity to the morphism $\text{Cur } \varphi$ of the corresponding current algebras.

5. THE OPERAD PERM

The operad Perm introduced in [7] is given by a family of spaces $\text{Perm}(n) = \mathbb{k}^n$ with natural composition rule

$$\gamma_{m_1, \dots, m_n} : e_k^{(n)} \otimes e_{j_1}^{(m_1)} \otimes \cdots \otimes e_{j_n}^{(m_n)} = e_{m_1 + \dots + m_{k-1} + j_k}^{(m_1 + \dots + m_n)},$$

where $e_k^{(n)}$, $k = 1, \dots, n$, is the standard basis of \mathbb{k}^n , $n \in \mathbb{N}$. Symmetric groups S_n act on $\text{Perm}(n)$ by permutations of coordinates.

Let \mathcal{P} be an operad. Denote by $\text{di}\mathcal{P}$ the Hadamard product $\mathcal{P} \otimes \text{Perm}$: $\text{di}\mathcal{P}(n) = \mathcal{P}(n) \otimes \text{Perm}(n)$, compositions and S_n -action are defined in the componentwise way.

Let us fix a cocommutative bialgebra H , and let ε stand for its counit. A left unital H -module C is in particular a linear space over the base field \mathbb{k} . For every $n \in \mathbb{N}$ consider \mathbb{k} -linear maps μ_n^k , $k = 1, \dots, n$, from $H^{\otimes n} \otimes_H C$ to C defined by

$$\mu_n^k : (h_1 \otimes \cdots \otimes h_n) \otimes_H c \mapsto \varepsilon(h_1 \overset{k}{\wedge} \cdots \wedge h_n) h_k c.$$

Lemma 1. *If (C, α) is a \mathcal{P} -conformal algebra then the family of maps $\{\alpha^{(0)}(n)\}_{n \in \mathbb{N}}$, $\alpha^{(0)}(n) : \text{di}\mathcal{P}(n) \rightarrow \mathcal{E}(C)(n)$, defined by*

$$(3) \quad \alpha^{(0)}(n)(f \otimes e_k^{(n)}) = \alpha(n)(f) \circ \mu_n^k,$$

$f \in \mathcal{P}(n)$, $k = 1, \dots, n$, $a_j \in C$, defines a morphism $\alpha^{(0)}$ of operads.

Proof. First, note that $\alpha^{(0)}(n)$ is S_n -linear. Indeed,

$$\alpha^{(0)}(n) : (f \otimes e_k^{(n)})^\sigma = f^\sigma \otimes e_{\sigma(k)}^{(n)} \mapsto \alpha(n)(f)^\sigma \circ \mu_n^{\sigma(k)} = (\alpha(n)(f) \circ \mu_n^k)^\sigma$$

since the action of σ on $\mathcal{E}^*(C)$ permutes the arguments of $\alpha(n)(f)$ together with tensor factors in $H^{\otimes n} \otimes_H C$, see [11].

Next, this is obvious that $\alpha^{(0)}(1)$ preserves the identity.

Finally, consider a composition $\gamma_{m_1, \dots, m_n}(f; g_1, \dots, g_n)$ in \mathcal{P} . By abuse of notations, assume

$$\alpha(m_l)(g_l) : a_1^{(l)} \otimes \dots \otimes a_{m_l}^{(l)} \mapsto F^{(l)} \otimes_H b^{(l)}, \quad F^{(l)} \in H^{\otimes m_l},$$

$l = 1, \dots, n$, and

$$\alpha(n)(f) : b^{(1)} \otimes \dots \otimes b^{(n)} \mapsto G \otimes_H c, \quad G \in H^{\otimes n},$$

$a_j^{(l)}, b^{(l)}, c \in C$. Then by definition

$$(4) \quad \alpha(m_1 + \dots + m_n)(\gamma_{m_1, \dots, m_n}(f; g_1, \dots, g_n)) : \\ a_1^{(1)} \otimes \dots \otimes a_{m_1}^{(1)} \otimes \dots \otimes a_1^{(n)} \otimes \dots \otimes a_{m_n}^{(n)} \\ \mapsto ((F^{(1)} \otimes \dots \otimes F^{(n)}) \otimes_H 1)(\Delta^{[m_1]} \otimes \dots \otimes \Delta^{[m_n]})G \otimes_H c,$$

where $\Delta^{[m]}(h) = \sum_{(h)} h_{(1)} \otimes \dots \otimes h_{(n)}$, $h \in H$.

Let us fix some $k \in \{1, \dots, n\}$, $j_l \in \{1, \dots, m_l\}$, $l = 1, \dots, n$. Suppose $F^{(l)} = h_1^{(l)} \otimes \dots \otimes h_{m_l}^{(l)}$, $G = h_1 \otimes \dots \otimes h_n$. Then by the properties of the counit ε the image of the right-hand side of (4) under $\mu_{m_1 + \dots + m_n}^{m_1 + \dots + m_{k-1} + j_k}$ is equal to

$$(5) \quad \varepsilon(F^{(1)}) \dots \varepsilon(F^{(n)}) \varepsilon(F_{j_k}^{(k)}) \varepsilon(G_k) h_{j_k}^{(k)} h_k c,$$

where $G_k = h_1 \overset{k}{\frown} h_n$ and $F_{j_k}^{(k)}$ is defined similarly.

On the other hand, let us compute the composition

$$\gamma_{m_1, \dots, m_n}(\alpha^{(0)}(n)(f \otimes e_k^{(n)}); \alpha^{(0)}(m_1)(g_1 \otimes e_{j_1}^{(m_1)}), \dots, \alpha^{(0)}(m_n)(g_n \otimes e_{j_n}^{(m_n)}))$$

in $\mathcal{E}(C)$. By (3), we have

$$\alpha^{(0)}(m_l)(g_l \otimes e_{j_l}^{(m_l)}) : a_1^{(l)} \otimes \dots \otimes a_{m_l}^{(l)} \mapsto \varepsilon(F_{j_l}^{(l)}) h_{j_l}^{(l)} b^{(l)}.$$

The $H^{\otimes n}$ -linearity of $\alpha(n)(f)$ implies

$$(6) \quad \alpha(n)(f) : \varepsilon(F_{j_1}^{(1)}) h_{j_1}^{(1)} b^{(1)} \otimes \dots \otimes \varepsilon(F_{j_n}^{(n)}) h_{j_n}^{(n)} b^{(n)} \\ \mapsto \varepsilon(F_{j_1}^{(1)}) \dots \varepsilon(F_{j_n}^{(n)}) (h_{j_1}^{(1)} \otimes \dots \otimes h_{j_n}^{(n)}) G \otimes_H c.$$

This is now obvious that the image of the right-hand side of (6) under μ_n^k coincides with (5). \square

Hence, every \mathcal{P} -conformal algebra (C, α) gives rise to a $\text{di}\mathcal{P}$ -algebra $C^{(0)} = (C, \alpha^{(0)})$. The correspondence $C \mapsto C^{(0)}$ is obviously a functor from the category of \mathcal{P} -conformal algebras to the category of $\text{di}\mathcal{P}$ -algebras.

6. DIALGEBRAS

Suppose \mathcal{P} is a quotient operad of \mathcal{F}_Ω , π is the corresponding morphism. This is easy to see that $\text{di}\mathcal{P}$ is a quotient of $\text{di}\mathcal{F}_\Omega = \mathcal{F}_\Omega \otimes \text{Perm}$ with respect to the morphism $\pi \otimes \text{id}$. In general, this is hard to determine the generators and defining relations of the Hadamard product of two operads, but due to the nice properties of Perm this is easy to do for $\text{di}\mathcal{P}$.

Consider the free operad $\mathcal{F}_{\Omega^{(2)}}$, where the language $\Omega^{(2)}$ is constructed in the following way. If $\Omega = \{f_i \mid i \in I\}$, $n_i = \nu(f_i)$, then $\Omega^{(2)} = \{f_i^k \mid i \in I, k = 1, \dots, n_i\}$, $\nu(f_i^k) = n_i$.

Define the morphism $\zeta_\Omega : \mathcal{F}_{\Omega^{(2)}} \rightarrow \text{di}\mathcal{F}_\Omega$ in the following way: $\zeta_\Omega(n)$ maps $f_i^k(x_1, \dots, x_{n_i}) \in \mathcal{F}_{\Omega^{(2)}}(n_i)$ to $f_i(x_1, \dots, x_{n_i}) \otimes e_k^{(n_i)}$. The composition of ζ_Ω with $\pi \otimes \text{id}$ provides a morphism $\pi^{(2)} : \mathcal{F}_{\Omega^{(2)}} \rightarrow \text{di}\mathcal{P}$.

Lemma 2. *For every $n \in \mathbb{N}$ the linear maps $\zeta_\Omega(n)$ and $\pi^{(2)}(n)$ are surjective.*

Proof. To prove the surjectivity of ζ_Ω (and hence of $\pi^{(2)}$) it is enough to show that $\text{di}\mathcal{F}_\Omega$ is generated by $f_i \otimes e_k^{(n_i)}$, $i \in I$, $k = 1, \dots, n_i$. In the binary case it was actually done in [18] and [11], the general case can be processed analogously. It is enough to construct a section $\rho(n) : (\mathcal{F}_\Omega \otimes \text{Perm})(n) \rightarrow \mathcal{F}_{\Omega^{(2)}}(n)$ such that $\rho(n) \circ \zeta_\Omega(n) = \text{id}$. It was done in [4], let us recall here the construction in terms of planar trees. Every monomial $f \in \mathcal{F}_\Omega(n)$ can be identified with a planar tree with n leaves (variables) labeled by numbers $1, \dots, n$ and vertices labeled by symbols from Ω , the degree (number of out-coming branches) of a vertex labeled by $f_i \in \Omega$ is equal to n_i . Then $f \otimes e_k^{(n)}$ may be considered as a tree with k th emphasized vertex. To get $\rho(n)(f \otimes e_k^{(n)})$ we should add superscripts to the labels of vertices in the tree corresponding to f in the following way. If a k th (counting from the left-hand side) out-coming branch of a vertex labeled by $f_i \in \Omega$ contains the emphasized leaf then the label is replaced with f_i^k . If neither of the out-coming branches in this vertex contain the emphasized leaf then the label is replaced with f_i^1 . \square

Suppose S is a set of polylinear polynomials such that the kernel of π is generated by S (e.g., if \mathcal{P} is a governing operad for a variety \mathfrak{P} then S consists of its defining identities). Consider the operad ideal $J(S)$ in $\mathcal{F}_{\Omega^{(2)}}$ generated by

$$(7) \quad f^k(x_1, \dots, x_{j-1}, g^l(x_j, \dots, x_{j+m-1}), x_{j+m}, \dots, x_{n+m-1}) \\ - f^k(x_1, \dots, x_{j-1}, g^p(x_j, \dots, x_{j+m-1}), x_{j+m}, \dots, x_{n+m-1}), \\ f, g \in \Omega, \quad n = \nu(f), \quad m = \nu(g), \quad k, j = 1, \dots, n, \quad k \neq j, \quad l, p = 1, \dots, m,$$

and

$$(8) \quad s^k(x_1, \dots, x_n), \quad s \in S \cap \mathcal{F}_\Omega(n), \quad n \in \mathbb{N}, \quad k = 1, \dots, n,$$

where $s^k = \rho(n)(s \otimes e_k^{(n)})$. Denote by $\mathcal{P}^{(2)}$ the quotient operad of $\mathcal{F}_{\Omega^{(2)}}$ with respect to $J(S)$, and let $\hat{\pi}^{(2)}$ be the corresponding morphism from $\mathcal{F}_{\Omega^{(2)}}$ to $\mathcal{P}^{(2)}$.

This is easy to see that $J(S)$ is contained in the kernel of $\pi^{(2)}$. Thus we have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{F}_{\Omega^{(2)}} & \xrightarrow{\zeta_\Omega} & \mathcal{F}_\Omega \otimes \text{Perm} & \xrightarrow{\text{pr}} & \mathcal{F}_\Omega \\ \hat{\pi}^{(2)} \downarrow & & \downarrow \pi \otimes \text{id} & & \downarrow \pi \\ \mathcal{P}^{(2)} & \longrightarrow & \mathcal{P} \otimes \text{Perm} & \xrightarrow{\text{pr}} & \mathcal{P} \end{array}$$

where pr stands for the natural projection.

Our aim is to show that the kernels of $\hat{\pi}^{(2)}$ and $\pi^{(2)}$ are equal.

Suppose (A, α) is a $\mathcal{P}^{(2)}$ -algebra. By abuse of notations, let us identify $f^k \in \mathcal{F}_{\Omega^{(2)}}(n)$ and their images under $\hat{\pi}^{(2)}(n)$ in $\mathcal{P}^{(2)}(n)$.

Let A_0 be the \mathbb{k} -linear span of all

$$\alpha(n)(f_i^p - f_i^l)(a_1, \dots, a_{n_i}), \quad i \in I, \quad a_j \in A, \quad p, l = 1, \dots, n_i.$$

It follows from (7) that A_0 is an ideal in the algebra A . Indeed, for every $f \in \Omega$

$$\alpha(n)(f^k)(a_1, \dots, a_{j-1}, b, a_{j+1}, \dots, a_n) = 0, \quad b \in A_0,$$

if $j \neq k$ ($f \in \Omega, n = \nu(f), k = 1, \dots, n$). For $j = k$, one may add $\alpha(n)(f^q)(a_1, \dots, a_{j-1}, b, a_{j+1}, \dots, a_n)$ with $q \neq k$ (which is zero) and make sure the result is again in A_0 .

Denote $\bar{A} = A/A_0$. The morphism α induces a morphism $\bar{\alpha} : \mathcal{P}^{(2)} \rightarrow \mathcal{E}(\bar{A})$, such that $(\bar{A}, \bar{\alpha})$ is the quotient $\mathcal{P}^{(2)}$ -algebra. In this algebra, the values of algebraic operations $\alpha(n)(f^k), f \in \Omega, n = \nu(f), k \in \{1, \dots, n\}$ do not depend on k , so this is actually an algebra of language Ω :

$$f^{\bar{A}}(\bar{a}_1, \dots, \bar{a}_n) = \overline{\alpha(n)(f^k)(a_1, \dots, a_n)}, \quad f \in \Omega, \quad n = \nu(f),$$

where $a_j \in A, \bar{a} = a + A_0 \in \bar{A}$. Moreover, it is obvious that \bar{A} is actually a \mathcal{P} -algebra.

Consider the formal direct sum of spaces $\hat{A} = \bar{A} \oplus A$ and define algebraic operations of language Ω on \hat{A} as follows:

$$g^{\hat{A}}(z_1, \dots, z_n) = \begin{cases} g^{\bar{A}}(z_1, \dots, z_n), & z_i \in \bar{A} \text{ for all } i = 1, \dots, n; \\ \alpha(n)(g^k)(a_1, \dots, a_n), & z_i = \bar{a}_i \in \bar{A} \text{ for all } i \neq k, \\ & z_k = a_k \in A; \\ 0, & \text{more than one } z_i \in A. \end{cases}$$

$g \in \Omega, \nu(g) = n$. Denote by $\hat{\alpha}$ the corresponding morphism from \mathcal{F}_{Ω} to $\mathcal{E}(\hat{A})$.

The definition of the canonical section ρ from Lemma 2 and induction on n imply that for every $s \in \mathcal{F}_{\Omega}(n)$, $s^{\hat{A}} := \hat{\alpha}(n)(s)$, we have

$$s^{\hat{A}}(z_1, \dots, z_n) = \rho(n)(s \otimes e_k^{(n)})(a_1, \dots, a_n) \in A \subseteq \hat{A}$$

if $z_i = \bar{a}_i \in \bar{A}$ for all $i \neq k$ and $z_k = a_k \in A$. Therefore, every $s \in S$ is an identity on \hat{A} , so $(\hat{A}, \hat{\alpha})$ is actually a \mathcal{P} -algebra.

Theorem 1. *The kernels of $\pi^{(2)}$ and $\hat{\pi}^{(2)}$ coincide, so the operads $\mathcal{P}^{(2)}$ and $\text{di}\mathcal{P}$ are equivalent.*

Proof. We have already seen that the kernel of $\hat{\pi}^{(2)}$ is contained in the kernel of $\pi^{(2)}$. Conversely, assume there exists an identity that holds on all $\text{di}\mathcal{P}$ -algebras but does not hold on some $\mathcal{P}^{(2)}$ -algebra (A, α) .

Consider the \mathcal{P} -algebra $(\hat{A}, \hat{\alpha})$ constructed above and fix a bialgebra H with a nonzero $T \in H$ such that $\varepsilon(T) = 0$. For example, one may consider the group algebra $H = \mathbb{k}\mathbb{Z}_2$.

The current conformal algebra $\text{Cur}\hat{A} = H \otimes \hat{A}$ is a \mathcal{P} -conformal algebra. By Lemma 1, $(\text{Cur}\hat{A})^{(0)}$ is a $\text{di}\mathcal{P}$ -algebra. Note that

$$A \rightarrow (\text{Cur}\hat{A})^{(0)}, \quad a \mapsto 1 \otimes \bar{a} + T \otimes a, \quad a \in A$$

is an injective homomorphism of $\Omega^{(2)}$ -algebras. Hence, A is in fact $\text{di}\mathcal{P}$ -algebra and thus satisfies all identities that hold on the class of such algebras. \square

Corollary 1. *Every $\text{di}\mathcal{P}$ -algebra A is embedded into $(\text{Cur}\hat{A})^{(0)}$ over an appropriate bialgebra H , \hat{A} is a \mathcal{P} -algebra.*

Lemma 3. *Consider $t = t(x_1, \dots, x_d) \in \text{di}\mathcal{P}(n)$. Then $t = t_1 \otimes e_1^{(n)} + \dots + t_n \otimes e_n^{(n)}$, $t_k \in \mathcal{P}(n)$. Let (A, α) be a \mathcal{P} -algebra. Then the dialgebra $(\text{Cur}A)^{(0)}$ satisfies the identity $t = 0$ if and only if A satisfies all identities $t_k = 0$, $k = 1, \dots, d$.*

Proof. It follows from the construction that if $\alpha(n)(t_k) = 0$ for all $k = 1, \dots, n$ then $(\text{Cur}\alpha)(n)(t_k) = 0$ and hence $(\text{Cur}\alpha)^{(0)}(n)(t) = 0$.

Conversely, consider $g = (\text{Cur}\alpha)^{(0)}(n)(t) \in \mathcal{E}(H \otimes A)(n)$ and compute

$$b_k = g(1 \otimes a_1, \dots, T \otimes a_k, \dots, 1 \otimes a_n), \quad k = 1, \dots, n,$$

for all $a_1, \dots, a_n \in A$. On the one hand, $b_k = 0$ since $(\text{Cur}A)^{(0)}$ satisfies the identity $t = 0$. On the other hand, $b_k = T \otimes \alpha(n)(t_k)(a_1, \dots, a_n)$, so A satisfies $t_k = 0$ for all k . \square

7. MORPHISMS OF OPERADS AND FUNCTORS

If \mathcal{P} and \mathcal{R} are two operads then every morphism $\alpha : \mathcal{P} \rightarrow \mathcal{R}$ gives rise to a functor from the category of \mathcal{R} -algebras to the category of \mathcal{P} -algebras. Namely, if (A, β) is an \mathcal{R} -algebra then the same space A with respect to the composition $\alpha \circ \beta : \mathcal{P} \rightarrow \mathcal{E}(A)$ is a \mathcal{P} -algebra. The correspondence $(A, \beta) \mapsto (A, \alpha \circ \beta)$ is obviously functorial. The construction that appears in Problem 1 is a particular case of such a functor.

Indeed, assume Ω and Ξ are two languages, \mathcal{F}_Ω and \mathcal{F}_Ξ are two corresponding free operads. Suppose $\pi : \mathcal{F}_\Omega \rightarrow \mathcal{P}$ and $\rho : \mathcal{F}_\Xi \rightarrow \mathcal{R}$ are two quotient morphisms to operads \mathcal{P} and \mathcal{R} governing some varieties of algebras.

Let $\omega : \mathcal{F}_\Omega \rightarrow \mathcal{F}_\Xi$ be a morphism of operads. The family of maps $\omega(n) : \mathcal{F}_\Omega(n) \rightarrow \mathcal{F}_\Xi(n)$ determines (and can be completely determined by) an interpretation of operations from Ω via operations from Ξ . We say that ω induces a morphism $\bar{\omega}\mathcal{P} \rightarrow \mathcal{R}$ iff $\text{Ker}\pi(n) \subseteq \text{Ker}(\omega(n) \circ \rho(n))$ for all $n \in \mathbb{N}$.

Example 1. Let Ω and Ξ contain one binary operation denoted by λ in Ω and μ in Ξ , then the morphism ω determined by the rule $\lambda = \mu - \mu^{(12)}$, $(12) \in S_2$, induces a morphism $\bar{\omega}$ from the operad Lie (governing the variety of Lie algebras) to the operad As (associative algebras). If (A, α) is an As -algebra then the pair $(A, \bar{\omega} \circ \alpha)$ is exactly the adjoint Lie-algebra of A (usually denoted by $A^{(-)}$).

The Hadamard product $\bar{\omega} \otimes \text{id} : \text{diLie} \rightarrow \text{diAs}$ defines the corresponding functor from the category of associative dialgebras to the category of Leibniz algebras (Lie dialgebras) [15].

A similar relation holds for Mal'cev dialgebras [6] and alternative dialgebras [13].

Example 2. Let JTS be the operad governing the variety of Jordan triple systems (see, e.g., [9]). Then there exists $\bar{\omega} : \text{JTS} \rightarrow \text{As}$ defined as follows: If $\tau = (\cdot, \cdot, \cdot) \in \text{JTS}(3)$ is the triple operation on JTS -algebras and $\mu \in \text{As}(2)$ is the product on associative algebras then $\bar{\omega}(\tau) = \gamma_{1,2}(\mu; \text{id}, \mu) + \gamma_{1,2}(\mu; \text{id}, \mu)^{(13)}$. This is the well-known construction of a Jordan triple system on an associative algebra: $(a, b, c) = abc + cba$.

In [4], the notion of a *Jordan triple disystem* (JTD) was introduced in such a way that $\text{JTD} = \text{JTS}^{(2)}$, in our notations. Theorem 1 immediately implies $\text{JTD} = \text{diJTS} = \text{JTS} \otimes \text{Perm}$. Hence, $\bar{\omega} \otimes \text{id} : \text{JTD} \rightarrow \text{diAs}$ defines a structure of a Jordan triple disystem on an associative dialgebra (c.f. [4, Theorem 5.10]).

Example 3. Let Jord stand for the operad governing the variety of Jordan algebras, $\mu \in \text{Jord}(2)$ is the commutative operation. Then there exists a morphism $\bar{\omega} : \text{JTS} \rightarrow \text{Jord}$ defined by $\bar{\omega}(\tau) = \gamma_{2,1}(\mu; \mu, \text{id}) - \gamma_{2,1}(\mu; \mu, \text{id})^{(23)} + \gamma_{1,2}(\mu; \text{id}, \mu)$, i.e., $(a, b, c) = (ab)c - (ac)b + a(bc)$.

The notion of a Jordan dialgebra was studied in [19, 3, 11], see also [5]. As in the previous example, Theorem 1 gives a new proof of the relation between Jordan dialgebras and Jordan triple disystems [4, Theorem 7.3].

Let us fix two quotient morphisms $\pi : \mathcal{F}_\Omega \rightarrow \mathcal{P}$, $\rho : \mathcal{F}_\Xi \rightarrow \mathcal{R}$, and a morphism $\omega : \mathcal{F}_\Omega \rightarrow \mathcal{F}_\Xi$ inducing a morphism from \mathcal{P} to \mathcal{R} which is also denoted by ω for simplicity.

Definition 1. A \mathcal{P} -algebra (A, α) is called ω -special if there exists an \mathcal{R} -algebra (A, β) such that $\alpha = \omega \circ \beta$. The same notion makes sense for conformal algebras over an arbitrary cocommutative bialgebra H .

Lemma 4. If (A, α) is an ω -special \mathcal{P} -algebra then $(\text{Cur } A, \text{Cur } \alpha)$ is an ω -special \mathcal{P} -conformal algebra.

Proof. Suppose there exists an \mathcal{R} -algebra (A, β) , $\beta : \mathcal{R} \rightarrow \mathcal{E}(A)$, such that $\alpha = \omega \circ \beta$. Then the claim follows from the observation

$$(9) \quad \omega \circ \text{Cur } \beta = \text{Cur } (\omega \circ \beta).$$

Indeed, for every $f \in \mathcal{P}(n)$ the pseudo-linear maps $(\omega \circ \text{Cur } \beta)(n)(f) \in \mathcal{E}^*(H \otimes A)$ are completely defined by their values at $(1 \otimes a_1, \dots, 1 \otimes a_n)$, $a_i \in A$. By the definition of Cur , we have $(\text{Cur } \beta)(n)(g)(1 \otimes a_1, \dots, 1 \otimes a_n) = (1 \otimes \dots \otimes 1) \otimes_H (\beta(n)(g))(a_1, \dots, a_n)$ for every $g \in \mathcal{R}(n)$, in particular, for $g = \omega(n)(f)$. This is now easy to see that left- and right-hand sides of (9) coincide at every $f \in \mathcal{P}(n)$. \square

Lemma 5. If (C, α) is an ω -special \mathcal{P} -conformal algebra then $(C, \alpha^{(0)})$ is an $(\omega \otimes \text{id})$ -special $\text{di}\mathcal{P}$ -algebra.

Proof. It is enough to show $(\omega \circ \beta)^{(0)} = (\omega \otimes \text{id}) \circ \beta^{(0)}$ for every $\beta : \mathcal{R} \rightarrow \mathcal{E}^*(C)$. Relation (3) implies

$$(\omega \circ \beta)^{(0)}(n) : f \otimes e_k^{(n)} \mapsto (\omega \circ \beta)(n)(f) \circ \mu_n^k = \beta(n)(\omega(n)(f)) \circ \mu_n^k$$

for every $f \in \mathcal{P}(n)$, $k = 1, \dots, n$. On the other hand,

$$((\omega \otimes \text{id}) \circ \beta^{(0)})(n) : f \otimes e_k^{(n)} \mapsto \beta^{(0)}(n)(\omega(n)(f) \otimes e_k^{(n)}) = \beta(n)(\omega(n)(f)) \circ \mu_n^k.$$

\square

The class of all ω -special \mathcal{P} -algebras is closed under Cartesian products. Therefore, the class of all homomorphic images of all subalgebras of ω -special \mathcal{P} -algebras is a variety \mathfrak{S} . Consider the set of polylinear identities that hold on this variety and define the corresponding operad $S^\omega \mathcal{P}$. This is a quotient operad of \mathcal{P} , and there exists a morphism $S^\omega : \mathcal{P} \rightarrow S^\omega \mathcal{P}$. Every \mathcal{P} -algebra from \mathfrak{S} is an $S^\omega \mathcal{P}$ -algebra, but the converse may not be true if $\text{char } \mathbb{k} > 0$.

Lemma 6. Consider the class of all quotient operads \mathcal{P}' of \mathcal{P} satisfying the following property: For every morphism $\alpha : \mathcal{R} \rightarrow \mathcal{E}(A)$ there exists a morphism

$\alpha' : \mathcal{P}' \rightarrow \mathcal{E}(A)$ such that the diagram

$$\begin{array}{ccccc} \mathcal{F}_\Omega & \xrightarrow{\pi} & \mathcal{P} & \xrightarrow{\pi'} & \mathcal{P}' \\ \downarrow & & \downarrow \omega & & \downarrow \alpha' \\ \mathcal{F}_\Xi & \xrightarrow{\rho} & \mathcal{R} & \xrightarrow{\alpha} & \mathcal{E}(A) \end{array}$$

is commutative. Then $S^\omega \mathcal{P}$ is a quotient of all such \mathcal{P}' .

Proof. Given $f \in \mathcal{P}(n)$, if $\pi'(n)(f) = 0$ then $(\omega \circ \alpha)(n)(f) = 0$ for every α . For the free countably generated \mathcal{R} -algebra (A, α) each $\alpha(n)$ is injective, so $\text{Ker } \pi'(n) \subseteq \text{Ker } \omega(n)$.

By the definition, $S^\omega \mathcal{P}$ satisfies the condition on \mathcal{P}' described above. Hence, the kernel of the quotient morphism $S^\omega : \mathcal{P} \rightarrow S^\omega \mathcal{P}$ contains all $f \in \mathcal{P}(n)$ such that $\omega(n)(f) = 0$ in $\mathcal{R}(n)$. Therefore, $\text{Ker } \pi'(n) \subseteq \text{Ker } \omega(n) \subseteq \text{Ker } S^\omega(n)$. \square

Corollary 2. *The kernel of $S^\omega : \mathcal{P} \rightarrow S^\omega \mathcal{P}$ coincides with the kernel of $\omega : \mathcal{P} \rightarrow \mathcal{R}$.*

8. SPECIALITY OF ALGEBRAS

In this section, we state the solution of Problems 1 and 2. First, let us reformulate the Problem 1 in a more general framework.

On the one hand, $\omega : \mathcal{F}_\Omega \rightarrow \mathcal{F}_\Xi$ gives rise to

$$\omega \otimes \text{id} : \text{di} \mathcal{F}_\Omega = \mathcal{F}_\Omega \otimes \text{Perm} \rightarrow \mathcal{F}_\Xi \otimes \text{Perm} = \text{di} \mathcal{F}_\Xi.$$

Thus, we may define $(\omega \otimes \text{id})$ -special $\text{di} \mathcal{P}$ -algebras and a variety $\mathfrak{S}^{(2)}$ generated by them. Consider the operad $S^{\omega \otimes \text{id}} \text{di} \mathcal{P}$ defined by all polylinear identities that hold on $\mathfrak{S}^{(2)}$. This is a generalization of the BSO procedure from [4]. In the case of zero characteristic, we can conclude that every $S^{\omega \otimes \text{id}} \text{di} \mathcal{P}$ -algebra is a homomorphic image of a subalgebra of an $(\omega \otimes \text{id})$ -special $\text{di} \mathcal{P}$ -algebra.

On the other hand, the variety of $S^\omega \mathcal{P}$ -algebras (defined by polylinear identities) gives rise to the corresponding variety of $\text{di} S^\omega \mathcal{P}$ -algebras, where $\text{di} S^\omega \mathcal{P} = S^\omega \mathcal{P} \otimes \text{Perm}$, as above.

Both operads $\text{di} S^\omega \mathcal{P}$ and $S^{\omega \otimes \text{id}} \text{di} \mathcal{P}$ are quotients of $\mathcal{P} \otimes \text{Perm}$ and, therefore, of $\mathcal{F}_\Omega \otimes \text{Perm}$.

In these terms, Problems 1 and 2 are particular cases of the following

Theorem 2. (1) *If $\text{char } \mathbb{k} = 0$ then $\text{di} S^\omega \mathcal{P} = S^{\omega \otimes \text{id}} \text{di} \mathcal{P}$.*

(2) *If $\text{char } \mathbb{k} = p > 0$ then the identities of degree $d < p$ that hold on $\text{di} S^\omega \mathcal{P}$ -algebras are the same that hold on $S^{\omega \otimes \text{id}} \text{di} \mathcal{P}$ -algebras*

Proof. (1) We will show that the class of $\text{di} S^\omega \mathcal{P}$ -algebras coincides with the class of $S^{\omega \otimes \text{id}} \text{di} \mathcal{P}$ -algebras.

" \supseteq ": To compare varieties, we may compare the sets of defining identities. Suppose $f = f(x_1, \dots, x_n) \in \mathcal{F}_{\Omega(2)}(n)$ is a polylinear identity that holds on the variety of all $\text{di} S^\omega \mathcal{P}$ -algebras. Hence, $\zeta_\Omega(n)(f) = \sum_{k=1}^n f_k \otimes e_k^{(n)} \in \text{Ker } (S^\omega \otimes \text{id})(n)$, so $f_k \in \text{Ker } S^\omega(n) = \text{Ker } \omega(n)$ for all $k = 1, \dots, n$, see Corollary 2. Therefore, $(\omega \otimes \text{id})(n)(\zeta_\Omega(n)(f)) = 0$, i.e., $\zeta_\Omega(n)(f)$ is an identity of the variety of all $S^{\omega \otimes \text{id}} \text{di} \mathcal{P}$ -algebras. Since the kernel of $\zeta_\Omega(n)$ annihilates in $\text{di} \mathcal{P}$, we obtain that f is an identity of the variety of all $S^{\omega \otimes \text{id}} \text{di} \mathcal{P}$ -algebras. Such a relation between identities implies the claim.

Note that this part of the proof does not depend on the characteristic of the base field.

" \subseteq ": Assume A is a $\text{di}S^\omega\mathcal{P}$ -algebra. Then by Corollary 1 $A \subseteq (\text{Cur } \hat{A})^{(0)}$, where \hat{A} is a $S^\omega\mathcal{P}$ -algebra. Hence, $\hat{A} = \varphi(B_1)$, where φ is a homomorphism of \mathcal{P} -algebras, $B_1 \subseteq B$, and B is an ω -special \mathcal{P} -algebra. Then $\text{Cur } B$ is an ω -special conformal \mathcal{P} -algebra by Lemma 4, therefore, $(\text{Cur } B)^{(0)}$ is an $(\omega \otimes \text{id})$ -special $\text{di}\mathcal{P}$ -algebra by Lemma 5. Since $(\text{Cur } B_1)^{(0)} \subseteq (\text{Cur } B)^{(0)}$ and $\text{Cur } \hat{A} = \text{Cur } \varphi(\text{Cur } B_1)$, we have $(\text{Cur } \hat{A})^{(0)} = (\text{Cur } \varphi)^{(0)}((\text{Cur } B_1)^{(0)})$, i.e., $(\text{Cur } \hat{A})^{(0)}$ is a homomorphic image of a subalgebra in an $(\omega \otimes \text{id})$ -special $\text{di}\mathcal{P}$ -algebra. Hence, A belongs to the variety of $S^{\omega \otimes \text{id}}\text{di}\mathcal{P}$ -algebras.

(2) We have to compare the sets of polylinear identities of degree $d < p$ that hold on all algebras from $\mathfrak{S}^{(2)}$ and on all $\text{di}S^\omega\mathcal{P}$ -algebras. Denote the first set by $\text{Id}_d(\mathfrak{S}^{(2)})$ and the latter one by $\text{Id}_d(\text{di}S^\omega\mathcal{P})$.

The embedding $\text{Id}_d(\text{di}S^\omega\mathcal{P}) \subseteq \text{Id}_d(\mathfrak{S}^{(2)})$ has already been proved in part (1). It remains to prove the converse.

Let $X = \{x_1, x_2, \dots\}$ be a countable set of variables and $\overline{X} = \{\bar{x}_1, \bar{x}_2, \dots\}$ be a copy of X . Consider the free $S^\omega\mathcal{P}$ -algebra $S^\omega\mathcal{P}\langle X \cup \overline{X} \rangle$ generated by X and \overline{X} . Since the class of $S^\omega\mathcal{P}$ -algebras is a variety defined by homogeneous identities, the notion of degree is well defined for its elements. Denote by $\deg_X f$ the degree of $f \in S^\omega\mathcal{P}\langle X \cup \overline{X} \rangle$ with respect to all elements from X .

Consider the $S^\omega\mathcal{P}$ -algebra

$$F(X) = S^\omega\mathcal{P}\langle X \cup \overline{X} \rangle / J,$$

where J is the linear span of all homogeneous $f \in S^\omega\mathcal{P}\langle X \cup \overline{X} \rangle$ such that $\deg_X f \geq 2$.

Assume $t = t(x_1, \dots, x_d) \in \text{Id}_d(\mathfrak{S}^{(2)})$. As an element of $\text{di}S^\omega\mathcal{P}(d)$, t can be identified with $t_1 \otimes e_1^{(d)} + \dots + t_d \otimes e_d^{(d)}$, $t_k \in S^\omega\mathcal{P}(d)$.

There exists a T-ideal Σ in $S^\omega\mathcal{P}\langle X \cup \overline{X} \rangle$ such that

$$S^\omega\mathcal{P}\langle X \cup \overline{X} \rangle / \Sigma \in \mathfrak{S}.$$

For every nonzero homogeneous $f \in \Sigma$ we have $\deg f \geq p$ since all identities of lower degree follow from polylinear identities that hold on $S^\omega\mathcal{P}$ -algebras by definition. Hence,

$$F_1(X) = S^\omega\mathcal{P}\langle X \cup \overline{X} \rangle / (J + \Sigma) \simeq F(X) / ((\Sigma + J) / J) \in \mathfrak{S},$$

so $(\text{Cur } F_1(X))^{(0)}$ satisfies the identity $t = 0$. By Lemma 3, $F_1(X)$ satisfies all identities $t_k = 0$, $k = 1, \dots, d$. But if $t_k \neq 0$ in $S^\omega\mathcal{P}(d)$ then $t_k(\bar{x}_1, \dots, \bar{x}_d) \notin \Sigma + J$ in $S^\omega\mathcal{P}\langle X \cup \overline{X} \rangle$ by the degree-related reasoning. Therefore, $t = 0$ in $\text{di}S^\omega\mathcal{P}(d)$. \square

Given a triple $(\mathcal{P}, \mathcal{R}, \omega)$ as above, a non-zero $f \in \mathcal{P}(n)$ is said to be a *special identity* if $f \in \text{Ker } \omega(n)$. As a corollary of Theorem 2, we may conclude that in all possible settings $(\text{di}\mathcal{P}, \text{di}\mathcal{R}, \omega \otimes \text{id})$ we should not expect an existence of polylinear special identities different from $f \otimes e_k^{(n)}$, where f is a special identity for $(\mathcal{P}, \mathcal{R}, \omega)$. This explains, in particular, the results of [6], [5], and [20] concerning special identities of Jordan and Mal'cev algebras.

To be more precise, state the following

Corollary 3. *Let $\text{char } \mathbb{k} = 0$. Suppose $f \in \mathcal{F}_{\Omega(2)}\langle X \rangle$, $X = \{x_1, x_2, \dots\}$, is a polynomial of type $\Omega(2)$. Denote by $L(f)$ the complete linearization of f , we may suppose $L(f) \in \mathcal{F}_{\Omega(2)}(n)$. Then the following conditions are equivalent.*

- (S1) $f = 0$ is an identity on all $(\omega \otimes \text{id})$ -special $\text{di}\mathcal{P}$ -algebras, but not an identity on all $\text{di}\mathcal{P}$ -algebras;
- (S2) $\zeta_\Omega(n)(L(f)) = \sum_{k=1}^n f_k \otimes e_k^{(n)}$, $f_k \in \mathcal{F}_\Omega$, where all f_k are identities on the class of all ω -special \mathcal{P} -algebras and at least one of them is not an identity on the class of all \mathcal{P} -algebras.

Proof. Over a field of characteristic zero, f is a special identity if and only if so is $L(f)$. If $L(f)$ does not hold on all $\text{di}\mathcal{P}$ -algebras then $(\pi(n) \otimes \text{id})\zeta_\Omega(n)(L(f)) \neq 0$, i.e., there exists k such that $\pi(n)(f_k) \neq 0$, so f_k does not hold on all \mathcal{P} -algebras. On the other hand, $(\omega(n) \otimes \text{id})\zeta_\Omega(n) = 0$, i.e., $\omega(n)(f_k) = 0$ for all k , so all f_k are identities on the class of all ω -special \mathcal{P} -algebras. The proof of the converse statement is similar. \square

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